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Asymptotic Properties of the Inelastic Kac Model

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We discuss the asymptotic behavior of certain models of dissipative systems obtained from a suitable modification of Kac caricature of a Maxwellian gas. It is shown that global equilibria different from concentration are possible if the energy is not finite. These equilibria are distributed like stable laws, and attract initial densities which belong to the normal domain of attraction. If the initial density is assumed of finite energy, with higher moments bounded, it is shown that the solution converges for large-time to a profile with power law tails. These tails are heavily dependent of the collision rule.

KEY WORDS: Granular gases; Boltzmann-like dissipative equations; long-time behavior of solutions.

1. INTRODUCTION

In recent times the study of the free and driven cooling of dissipative granular gases has received a lot of attention. Essential progresses have been made for simplified models, in particular thanks to the consideration of the Boltzmann equation for inelastic Maxwell particles, both for the free case without energy input,[2, 4] and for the driven case.[5, 11, 13] Inelastic Maxwell models share with elastic Maxwell molecules the property that the collision rate in the Boltzmann equation is independent of the relative velocity of the colliding pair. In granular gases however it is usual to consider Boltzmann-like equations for partially inelastic hard spheres. This choice relies on the physical hypothesis that the grains must be cohesionless, which implies the hard-sphere interaction only, and no long-range forces of any kind. Hence, inelastic Maxwell models do not describe real particles, but only pseudo-particles with collision rules between pre- and post-collisional velocities,

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defined to be the same as for the partially inelastic hard spheres with a constant coefficient of restitution. These models are of interest for granular fluids in spatially homogeneous states because of the mathematical simplifications resulting from a velocity independent collision rate. Among others properties, it is remarkable that the inelastic Maxwell models exhibit similarity solutions. Although it is not too difficult to describe possible classes of these solutions, it is in general quite difficult to show that these solutions represent the intermediate asymptotic of a wide class of initial conditions.

In kinetic theory of rarefied gases, the first results of this type were obtained by Bobylev and Cercignani[5, 6] for similarity solutions to the elastic Boltzmann equation characterized by the atypical property to have infinite energy. It was shown that these solutions describe a large-time asymptotic of a class of initial distributions. The study of the analogous problem for the inelastic Maxwell models leaded to stronger results. Ernst and Brito[15, 16] studied the large-time behavior of solutions and conjectured that these solutions must have a self-similar asymptotic for a wide class of initial data. The first proof of Ernst–Brito conjecture for a sub-class of isotropic initial conditions was obtained in ref. 8. The (unphysical) restriction to the initial condition was subsequently removed in ref. 9, where it was proven that the self-similar solution attracts all data which initially have finite moments of some order greater than two (the energy).

On the contrary to elastic collisions, partially inelastic collisions have a nontrivial outcome as well in one dimension, and the one-dimensional idealization is a nontrivial adjunct to more realistic studies. The choice of a one-dimensional Boltzmann equation represents in general a compromise between the requirement to have a good approximation of dilute granular systems from the physical point of view, and reasonable difficulties of solutions both from the mathematical and numerical point of views. Among others, a one-dimensional model of the Boltzmann equation for hard-spheres performing partially inelastic collisions, where the coefficient of restitution depends on the relative speed, has been recently studied in ref. 25.

One-dimensional pseudo-Maxwellian inelastic gases where studied in ref. 1. This study led to the discovery of an exact similarity solution for a freely cooling pseudo-Maxwellian inelastic gas.[1] This solution, which has an algebraic high energy tail like $1/v^4$, can be used to test the class of initial values that are attracted in large-time.

In this paper we introduce and study a one-dimensional dissipative kinetic model which is sufficiently rich to exhibit a variety of steady states and similarity solutions. The model can be viewed as a dissipative generalization of the Kac caricature of a Maxwell gas introduced in the fifties.[23]

Kac model has been fruitfully used from that time on, to find explicit rates of convergence towards the Maxwellian equilibrium,[24, 27] since its simple structure (with respect to the full Boltzmann equation) makes possible to carry out exact computations. We introduce and discuss the inelastic Kac model together with its representation in Fourier space, in Sections 2 and 3. Here we will in addition discuss the existence of non Maxwellian equilibria, showing that these equilibria have infinite energy. Section 4 illustrates how the problem we are dealing with is deeply connected with the central limit theorem for stable laws, like the classical central limit theorem is closely connected with the convergence towards equilibrium of the elastic Boltzmann or Kac equation. Having in mind this connection, in Sections 5 and 6 we obtain both uniqueness and convergence results in terms of a distance which has been used recently in connection with the Boltzmann equation[17, 26] and the central limit theorem for stable laws[18] as well. The main result is that these Maxwellian equilibria attract any initial data which is in a suitable (small) domain of attraction. The second part of the paper deals with the existence of exact (similarity) solutions, and to the proof of Ernst–Brito conjecture, that is to the proof of the convergence of the solution towards a steady state. In particular, we will discuss the importance of the inelasticity rule to get the tails of these steady states. The main result here is that the rate of convergence depends on the number of finite moments of the steady state itself. For the case one can handle explicitly, that is for the self-similar solution found in ref. 1, that has all moments bounded of order $2+\alpha$, with $0 \leqslant \alpha < 1$, one shows that this solution represents the intermediate asymptotic of any other solution which has $2+\alpha$ moments bounded, with an explicit rate of convergence which can be evaluated in function of the maximum order of moments of the initial datum. The results here show that the situation with self-similar asymptotics is very close to the case of the multidimensional inelastic Maxwell model studied in ref. 9. The last section is devoted to possible extensions and applications of the techniques.

2. THE INELASTIC KAC MODEL

We shall consider here a Boltzmann-type equation for a granular gas subject to dissipative collisions and only for space independent data. This simple one-dimensional kinetic model, which can be viewed as the dissipative generalization of the Kac caricature of a Maxwellian gas,[23] reads

$$\frac{\partial f}{\partial t}(v, t) = Q_p(f, f)(v, t) \tag{1}$$

where the right-hand side of (1) describes the rate of change of the density function f due to dissipative collisions,

$$Q_p(f,f)(v) = \int_{\mathbb{R}\times[0,2\pi]} \frac{d\theta}{2\pi}\, dw[\chi^{-1} f(v_p^{**})\, f(w_p^{**}) - f(v)\, f(w)]. \tag{2}$$

The velocities (v_p^{**}, w_p^{**}) are the pre collisional velocities of the so-called inverse collision, which results with (v, w) as post collisional velocities. Given (v, w), the post collisional velocities (v_p^*, w_p^*) are defined simply generalizing the Kac rule

$$\begin{aligned} v_p^* &= v \cos\theta\, |\cos\theta|^p - w \sin\theta\, |\sin\theta|^p \\ w_p^* &= v \sin\theta\, |\sin\theta|^p + v \cos\theta\, |\cos\theta|^p. \end{aligned} \tag{3}$$

In (3) the positive constant $p < +\infty$ measures the degree of inelasticity. If $p = 0$, the binary collision is elastic and we obtain the classical Kac equation, where the post collisional velocities are given by a rotation in the (v, w) plane. The factor $\chi = |\sin\theta|^{2+2p} + |\cos\theta|^{2+2p}$ in the gain term appears respectively from the Jacobian of the transformation $dv^{**}\, dw^{**}$ into $dv\, dw$. The lost of energy in a single binary collision depends on the choice of the *inelasticity parameter* p, and it is given by

$$(v_p^*)^2 + (w_p^*)^2 = (v^2 + w^2)(|\sin\theta|^{2+2p} + |\cos\theta|^{2+2p}). \tag{4}$$

The structure of the inelastic Kac equation is similar to the inelastic Boltzmann equation for a Maxwell gas, and also here mass is conserved, while energy is nonincreasing. However, momentum is not conserved unless it is zero initially, and therefore this will be the only case considered. Moreover, the Kac equation does not obey an H-theorem. Here normalizations are chosen so that

$$\int_{\mathbb{R}} f(v,0)\, dv = 1; \qquad \int_{\mathbb{R}} v f(v,0)\, dv = 0; \qquad \int_{\mathbb{R}} v^2 f(v,0)\, dv = 1. \tag{5}$$

One of the important properties of Maxwell models is that the moment equations form a set of closed equations. Indeed, given any function $\varphi = \varphi(v)$, the evolution of $\langle\varphi\rangle$ is given by

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \varphi(v)\, f(v,t)\, dv \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2\times[0,2\pi]} \frac{d\theta}{2\pi}\, dv\, dw[\varphi(v_p^*) + \varphi(w_p^*) - \varphi(v) - \varphi(w)]\, f(v)\, f(w). \end{aligned} \tag{6}$$

Choosing $\varphi(v) = v^2$ shows that the average kinetic energy of granular temperature $T(t) = \langle v^2 \rangle$ keeps decreasing at a rate proportional to the inelasticity

$$\gamma_p = 1 - \int_0^{2\pi} \frac{d\theta}{2\pi} \left(|\sin\theta|^{2+2p} + |\cos\theta|^{2+2p}\right). \tag{7}$$

Thus, if the initial density has finite temperature, the solution to the inelastic Kac equation does not reach the Maxwellian $\omega(v) = (2\pi)^{-1/2} \exp[-v^2/2]$, but is approaching a Dirac delta function $\delta(v)$ for large times. It is well-known[16] that the rate of decay of the energy in Maxwell models does not satisfy Haff's law,[19] which predicts a decay like t^{-2} for a gas composed of inelastic hard spheres. However this rate of decay can be easily obtained for Maxwell models by rescaling the time (see the discussion in ref. 16).

3. THE FOURIER TRANSFORM OF KAC EQUATION AND STEADY STATES OF INFINITE ENERGY

The Kac model (1) can be studied in weak form

$$\frac{d}{dt} \int \varphi(v)\, f(v)\, dv = \int_{\mathbb{R}\times\mathbb{R}\times[0,2\pi]} \frac{d\theta}{2\pi} \{\varphi(v_p^*) - \varphi(v)\}\, f(v)\, f(w)\, dv\, dw$$

we shall study this with the normalization conditions (5). It is equivalent to use the Fourier transform of the equation:[3]

$$\frac{\partial \hat{f}(\xi, t)}{\partial t} = \hat{Q}_p(\hat{f}, \hat{f})(\xi, t), \tag{8}$$

where $\hat{f}(\xi, t)$ is the Fourier transform of $f(x, t)$,

$$\hat{f}(\xi, t) = \int_{\mathbb{R}} e^{-i\xi v}\, f(v, t)\, dv,$$

and

$$\hat{Q}_p(\hat{f}, \hat{f})(\xi) = \int_0^{2\pi} \frac{d\theta}{2\pi} [\hat{f}(\xi_p^+)\, \hat{f}(\xi_p^-) - \hat{f}(\xi)\, \hat{f}(0)]. \tag{9}$$

In (9)

$$\begin{cases} \xi_p^+ = \xi \cos\theta\, |\cos\theta|^p \\ \xi_p^- = \xi \sin\theta\, |\sin\theta|^p, \end{cases} \tag{10}$$

and the initial conditions (5) turn into

$$\hat{f}(0) = 1, \qquad \hat{f}'(0) = 0, \qquad \hat{f}''(0) = -1,$$

$\hat{f} \in C^2(\mathbb{R}^d)$. Hence Eq. (8) can be rewritten as

$$\frac{\partial \hat{f}(\xi, t)}{\partial t} + \hat{f}(\xi, t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \hat{f}(\xi_p^+)\, \hat{f}(\xi_p^-). \tag{11}$$

Note that

$$|\xi_p^+|^{2/(1+p)} + |\xi_p^-|^{2/(1+p)} = |\xi|^{2/(1+p)}. \tag{12}$$

It is immediate to split (11) into a system of equations for the real and imaginary parts of $\hat{f}$. Thus we write $\hat{f}(\xi, t) = \phi(\xi, t) + i\psi(\xi, t)$, and obtain for the real part

$$\frac{\partial \phi(\xi, t)}{\partial t} + \phi(\xi, t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \phi(\xi_p^+)\, \phi(\xi_p^-), \tag{13}$$

with

$$\phi(\xi, 0) = \operatorname{Re} \hat{f}(\xi, 0)$$

and for the imaginary part

$$\frac{\partial \psi(\xi, t)}{\partial t} + \psi(\xi, t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \psi(\xi_p^+)\, \phi(\xi_p^-), \tag{14}$$

with

$$\psi(\xi, 0) = \operatorname{Im} \hat{f}(\xi, 0).$$

The interesting aspect of this splitting, as observed in ref. 17 for the elastic Kac equation, is that the real part (which corresponds to the even part of f) satisfies the same equation as $\hat{f}$, while in the equation for ψ (which corresponds to the odd part of f) the right-hand side vanishes. Thus

$$\psi(\xi, t) = \psi_0(\xi)\, e^{-t}, \tag{15}$$

and the same holds for the odd part of the solution f. One would like then to obtain a similar result for the real part.

If we leave the usual hypothesis of solutions having finite energy, it is immediate to show that the dissipative Kac equation admits nontrivial steady states. In fact, in consequence of equality (12), the collision operator (9) vanishes if $f(v) = M_p(v)$, where

$$\hat{M}_p(\xi) = \exp\{-\sigma\, |\xi|^{\frac{2}{1+p}}\}, \qquad \sigma > 0. \tag{16}$$

The function (16) is nothing but the characteristic function of a stable distribution of exponent $s = 2/(1+p)$, where $0 < s < 2$.[21, 22]

It is well-known that a Maxwellian is a strictly stable distribution (corresponding to $p = 0$), while the Cauchy law is 1-stable.[22] This enlightens a remarkable difference between the problem of convergence towards equilibrium for the elastic and the inelastic Kac models, the former being the analogous of the classical central limit theorem, while the latter is the analogous of the central limit theorem for stable laws. Having in mind this analogy, it becomes clear that, while in the elastic case convergence towards the Maxwellian is proven under weak assumptions on the initial data (essentially finite moments of order strictly greater than two),[17] in the inelastic case convergence towards the steady state can be proven only if the initial data belong to the so-called normal domain of attraction.[21, 22]

4. CENTRAL LIMIT THEOREM FOR STABLE LAWS AND DISSIPATION OF ENERGY

We illustrate in this section the analogies between the convergence towards a stable law in the central limit theorem[22] and the large-time asymptotic of the dissipative Kac equation. The central limit theorem for a centered p-stable law with distribution θ consists of finding the set of distributions F such that, when dealing with the normalized sum

$$S_n = \frac{X_1 + \cdots + X_n}{n^{1/s}}$$

of the independent and identically distributed random variables X_i with common distribution function F, the distribution F_n of the sum S_n converges to θ. Let us denote with $f(x)$, $x \in \mathbb{R}$, the probability density function of the random variables X_i, where f is normalized to satisfy 5, and with $f_n(x)$ the density of S_{2^n}, $n \geqslant 1$. Then, since

$$S_{2^{n+1}} = \frac{1}{2^{1/s}}\, S_{2^n} + \frac{1}{2^{1/s}}\, S^*_{2^n},$$

where S_{2^n} and $S^*_{2^n}$ are independent and identically distributed,

$$f_{n+1}(x) = \int_{\mathbb{R}} dy\, 2^{1/s} f_n(2^{1/s}(x-y))\, 2^{1/s} f_n(2^{1/s} y). \tag{17}$$

Let us change variable into the integral in (17), setting $x-y=\frac{1}{2}(x+z)$ (which implies $y=\frac{1}{2}(x-z)$). We obtain

$$f_{n+1}(x)=\int_{\mathbb{R}} dz\, 2^{\frac{2-s}{s}} f_n\left(\frac{x+z}{2^{\frac{s-1}{s}}}\right) f_n\left(\frac{x-z}{2^{\frac{s-1}{s}}}\right). \tag{18}$$

Since $f_n(x)$ has unit mass, we can rewrite (17) as

$$f_{n+1}(x)=f_n(x)+\int_{\mathbb{R}} dz\left\{2^{\frac{2-s}{s}} f_n\left(\frac{x+z}{2^{\frac{s-1}{s}}}\right) f_n\left(\frac{x-z}{2^{\frac{s-1}{s}}}\right)-f_n(x)\, f_n(z)\right\}. \tag{19}$$

The recursive relation (19) can be viewed as the explicit Euler scheme (at discrete times $\Delta t=1$) of the kinetic equation

$$\frac{\partial f}{\partial t}(x,t)=Q_s^*(f,f)(x,t), \tag{20}$$

where

$$Q_s^*(f,f)(x)=\int_{\mathbb{R}} dz\{2^{\frac{2-s}{s}} f(x_s^{**})\, f(z_s^{**})-f(x)\, f(z)\}. \tag{21}$$

In (21) (x_s^{**}, z_s^{**}) are the pre-collisional velocities in the *collision* having (x, z) as post-collisional velocities. Hence, the post-collisional velocities (x_s^*, z_s^*) are related to (x, z) by the relations

$$x_s^*=\frac{1}{2^s}(x+z) \tag{22}$$

$$z_s^*=\frac{1}{2^s}(x-z). \tag{23}$$

Since for $s<2$

$$(x_s^*)^2+(z_s^*)^2=2^{\frac{s-2}{s}}(x^2+z^2)<x^2+z^2,$$

the operator is mass preserving while it dissipates the second moment (energy).

This clarifies the deep analogies between the classical problem of the convergence towards a stable law, and the cooling problem in kinetic theory of dissipative gases. From this analogy it follows at once that methods developed for the former problem can be fruitfully applied to the latter. Recently, the rate of convergence towards a stable law in the central limit theorem has been investigated by means of a probability metric which allows for explicit computations.[18] The importance of this metric in the analysis of the convergence towards equilibrium for the elastic Kac equation has been remarked in ref. 17. We introduce and discuss this metric in the forthcoming section.

5. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Denote by $\mathscr{P}(\mathbb{R})$ the class of all probability distributions, and with $\mathscr{P}_s(\mathbb{R})$, $s > 0$, the class of all probability distributions $F \in \mathscr{P}$ such that

$$\int_{\mathbb{R}} |v|^s \, dF(v) < \infty.$$

We introduce a metric on $\mathscr{P}_s(\mathbb{R}^d)$ by

$$d_s(F, G) = \sup_{\xi \in \mathbb{R}} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s} \tag{24}$$

where $\hat{f}$ is the Fourier transform of F,

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi v} \, dF(v).$$

Let us write $s = m + \alpha$, where m is an integer and $0 \leqslant \alpha < 1$. In order that $d_s(F, G)$ be finite, it suffices that F and G have the same moments up to order m.

The norm (24) has been introduced in ref. 17 to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. There, the case $s = 2 + \alpha$, $\alpha > 0$, was considered. Further applications of d_s, with $s = 4$, were studied in ref. 10, while the cases $s = 2$ and $s = 2 + \alpha$, $\alpha > 0$, have been considered in ref. 12 in connection with the so-called Mc Kean graphs.[24] The case $s = 2$ was subsequently used in ref. 26, in connection with the uniqueness of the non cut-off Boltzmann equation for Maxwell molecules. A further application of the general case $s > 0$ to the finding of Berry–Essen type bounds in the central limit theorem for a stable law has been given in ref. 18. In this paper, we shall be interested mainly in this last case, which presents several analogies with the problem under study.

The existence of a solution to Eq. (1) can be seen easily using the same methods available for the elastic Kac model. In particular, a solution can be expressed as a Wild sum.[3, 10] Let f and g be two solutions of the Kac equation (1), and $\hat{f}$, $\hat{g}$ their Fourier transforms. Then, given any positive constant s,

$$\frac{\partial}{\partial t}\frac{(\hat{f}-\hat{g})}{|\xi|^s}+\frac{\hat{f}(\xi)-\hat{g}(\xi)}{|\xi|^s}=\int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{\hat{f}(\xi_p^+)\,\hat{f}(\xi_p^-)-\hat{g}(\xi_p^+)\,\hat{g}(\xi_p^-)}{|\xi|^s}. \tag{25}$$

Now, we do the usual splitting

$$\begin{aligned}
&\left|\frac{\hat{f}(\xi_p^+)\,\hat{f}(\xi_p^-)-\hat{g}(\xi_p^+)\,\hat{g}(\xi_p^-)}{|\xi|^s}\right| \\
&\quad\leqslant |\hat{f}(\xi_p^+)|\left|\frac{\hat{f}(\xi_p^-)-\hat{g}(\xi_p^-)}{|\xi_p^-|^s}\right|\frac{|\xi_p^-|^s}{|\xi|^s}+|\hat{g}(\xi_p^-)|\left|\frac{\hat{f}(\xi_p^+)-\hat{g}(\xi_p^+)}{|\xi_p^+|^s}\right|\frac{|\xi^+|^s}{|\xi|^s} \\
&\quad\leqslant \sup\left|\frac{\hat{f}-\hat{g}}{|\xi|^s}\right|\left(\frac{|\xi_p^-|^s+|\xi_p^+|^s}{|\xi|^s}\right).
\end{aligned}$$

Owing to condition (12), if we choose $s=2/(1+p)$ we obtain

$$\left|\frac{\hat{f}(\xi_p^+)\,\hat{f}(\xi_p^-)-\hat{g}(\xi_p^+)\,\hat{g}(\xi_p^-)}{|\xi|^{2/(1+p)}}\right|\leqslant \sup\left|\frac{\hat{f}-\hat{g}}{|\xi|^{2/(1+p)}}\right|. \tag{26}$$

We set

$$h(t,\xi)=\frac{\hat{f}(\xi)-\hat{g}(\xi)}{|\xi|^{2/(1+p)}}.$$

The preceding computation shows that

$$\frac{\partial h}{\partial t}\leqslant(\|h\|_\infty-h). \tag{27}$$

Gronwall's lemma proves at once that $\|h(t)\|_\infty$ is non increasing. We have

Theorem 5.1. Let $f(t)$ and $g(t)$ be two solutions of the Kac equation (1), Then, if $s\geqslant 2/(1+p)$, for all times $t\geqslant 0$,

$$d_s(f(t),g(t))\leqslant d_s(f(0),g(0)).$$

In particular, let f_0 be a nonnegative density with finite moment of order $s=2/(1+p)$. Then, there exists a unique weak solution $f(t)$ of the Kac equation, such that $f(0)=f_0$.

6. CONVERGENCE TO EQUILIBRIUM

In the central limit problem, the normal domain of attraction (NDA) of a s-stable law consists of the set of distributions F such that, when dealing with the normalized sum

$$S_n = \frac{X_1 + \cdots + X_n}{n^{1/s}}$$

of the independent and identically distributed random variables X_i with common distribution function F, S_n converges to M_s. Hence, the normal domain of attraction is characterized by requiring that, for all $\xi \in N$,

$$\lim_{n \to \infty} n(\hat{f}(\xi/n^{1/s}) - 1) = -\sigma\, |\xi|^s.$$

Equivalent definitions of the NDA in terms of properties of the distribution function can be found in ref. 22.

As in the classical problem for a normal law, informations on the rate of convergence and restriction of the set of NDA (in order to get a rate of convergence) depend on the choice of the metric. In terms of the Fourier-based metric defined by (24), it is natural to introduce the set

$$D_{\sigma, s} = \left\{ f \in \mathscr{P},\ \hat{f}(\xi) = 1 - \sigma\, |\xi|^s + \psi_f(\xi), \text{ with } \frac{\psi_f(\xi)}{|\xi|^s} \in L^\infty(\mathbb{R}) \right\}. \tag{28}$$

The importance of such a set for the convergence towards equilibrium for the inelastic Kac model is easily understood if one realizes that in the elastic case $D_{\sigma,2}$ contains all densities with bounded second moment, with a precise control of the remainder ψ_f, which has to decay faster than $|\xi|^2$.[17]

It is clear that the $d_s(f, g)$ is finite when f and g belong to the set $D_{\sigma, s}$. Notice also that the stable steady state M_s belongs to $D_{\sigma, s}$ (and actually the remainder goes to 0 faster than $|\xi|^s$).

Following ref. 18, the convergence problem towards equilibrium, will be discussed by introducing the Fourier domain of attraction (FDA)

$$D^0_{\sigma, s} = \left\{ f \in D_{\sigma, s},\ \frac{\psi(\xi)}{|\xi|^s} \xrightarrow[\xi \to 0]{} 0 \right\}$$

and a suitable subset of the set $D^0_{\sigma, p}$. To this extent, we consider, for $\delta > 0$, the set

$$D^\delta_{\sigma, s} = \{ f \in D^0_{\sigma, s},\ |\psi(\xi)|/|\xi|^s \leqslant |\xi|^\delta \}.$$

Notice that, with $\hat{M}_s(\xi) = e^{-\sigma|\xi|^s}$ and $f \in D^\delta_{\sigma,s}$, one has $d_{s+\delta}(f, M_s) \leqslant C < \infty$.

A easy-to-check criterion ensuring that a given density f lies in $D^\delta_{\sigma,s}$, relies on finiteness of some pseudo-moment. For the normal case $s=2$, M_s has all moments finite, so that integrability of $x^\alpha(f - M_s)$ is equivalent to integrability for $x^\alpha f$. This of course does not apply when $s<2$. Note also that, for $s=2$, the set considered in ref. 17 coincides with

$$\left\{ f \in \mathscr{P}, \int |v|^{2+\delta} f(v)\, dv < \infty \right\} \subset D^\delta_{\sigma,2}$$

with a strict inclusion. We report here the following result obtained in ref. 18,

Lemma 6.1. Let $0 \leqslant s < 1$ and $0 < \delta \leqslant 1-s$. Then, the following embedding holds

$$\left\{ f \in \mathscr{P}, \int |v|^{s+\delta} |f(v) - M_s(v)|\, dv < \infty \right\} \subset D^\delta_{\sigma,s}.$$

Let $1 \leqslant s < 2$ and $0 \leqslant \delta \leqslant 2-s$. Then

$$\left\{ f \in \mathscr{P}, \int |v|^{s+\delta} |f(v) - M_s(v)|\, dv < \infty, \int v\,(f(v) - M_s(v))\, dv = 0 \right\} \subset D^\delta_{\sigma,s}.$$

We prove the main result of this section.

Theorem 6.2. Let $p>1$, and let $f(t)$ be the unique solution of the Kac equation (1), corresponding to the initial density f_0 such that, for some $0 < \delta \leqslant \frac{p-1}{p+1}$

$$\int |v|^{s+\delta} |f_0(v) - M_s(v)|\, dv < \infty,$$

where $s = 2/(1+p)$. Then, $f(t)$ converges exponentially fast in Fourier metric towards equilibrium, and the following bound holds

$$d_{s+\delta}(f(t), M_s) \leqslant d_{s+\delta}(f_0, M_s) \exp\{-(1 - A_{p,\delta})\, t\} \tag{29}$$

where

$$A_{p,\delta} = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(|\sin\theta|^{2+(1+p)\delta} + |\cos\theta|^{2+(1+p)\delta}\right) < 1.$$

Let now $0 < p \leqslant 1$, and let $f(t)$ be the unique solution of the Kac equation (1), corresponding to the initial density f_0 such that, for some $0 < \delta \leqslant \frac{2p}{p+1}$

$$\int |v|^{s+\delta} |f_0(v) - M_s(v)| \, dv < \infty.$$

Then, $f(t)$ converges exponentially fast in Fourier metric towards equilibrium, and bound (29) holds.

Proof. The proof is an easy consequence of the computations leading to Theorem 5.1. Thanks to Lemma 6.1, the hypotheses on the initial value f_0 are such that $\hat{f}_0 \in D^{\delta}_{\sigma, s}$, so that $d_{s+\delta}(f_0, M_s)$ is bounded. Given $\delta > 0$, proceeding as before we get the bound

$$\left| \frac{\hat{f}(\xi_p^+)\, \hat{f}(\xi_p^-) - \hat{g}(\xi_p^+)\, \hat{g}(\xi_p^-)}{|\xi|^{s+\delta}} \right| \leqslant \sup \left| \frac{\hat{f} - \hat{g}}{|\xi|^{s+\delta}} \right| \left(\frac{|\xi_p^-|^{s+\delta} + |\xi_p^+|^{s+\delta}}{|\xi|^{s+\delta}} \right). \tag{30}$$

Using the definitions of ξ_p^-, ξ_p^+, one obtains

$$\frac{|\xi_p^-|^{s+\delta} + |\xi_p^+|^{s+\delta}}{|\xi|^{s+\delta}} = |\sin \theta|^{2+(1+p)\delta} + |\cos \theta|^{2+(1+p)\delta}.$$

Hence, from (25) follows

$$\left| \frac{\partial}{\partial t} \frac{(\hat{f} - \hat{g})}{|\xi|^{s+\delta}} + \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^{s+\delta}} \right| \leqslant A_{p,\delta} \left\| \frac{\hat{f}(\xi) - \hat{g}(\xi)}{|\xi|^{s+\delta}} \right\|_{\infty} \tag{31}$$

Gronwall's lemma then proves (29). ∎

7. SIMILARITY SOLUTIONS

In the previous section we discussed, in connection with the freely cooling without energy input of the dissipative Kac equation, the role of the steady states (16). Our main result states that these Maxwellians represent the large-time asymptotic of any density which lies initially in the corresponding domain of attraction, and that the convergence is exponentially fast. Here, we will investigate the existence of exact solutions, and their importance in the problem of cooling of the dissipative Kac equation. To this aim, given $p > 0$, we look for solutions to Eq. (1) for the density $g(v, t)$, where g is defined through f via the scaling

$$g(v, t) = e^{-\alpha t} f(e^{-\alpha t} v, t), \qquad \alpha > 0. \tag{32}$$

Let us point out that thanks to its definition, when $\alpha = \gamma_p/2$, $e^{-\alpha t} = \sqrt{T(t)}$, and

$$\int_{\mathbb{R}} v^2 g(v, t)\, dv = 1, \tag{33}$$

i.e., the granular temperature of $g(v, t)$ does not change with time.

We deduce now the evolution equation for the rescaled distribution function g. To simplify notations, let us set, for any $p > 0$, $\gamma = \gamma_p$, $\tilde{v} = e^{\alpha t} v$. We have

$$\begin{aligned} \frac{\partial f}{\partial t}(v, t) &= \alpha e^{\alpha t} g(\tilde{v}, t) + e^{\alpha t} \alpha \tilde{v} \frac{\partial g}{\partial \tilde{v}}(\tilde{v}, t) + e^{\alpha t} \frac{\partial g}{\partial t}(\tilde{v}, t) \\ &= e^{\alpha t} \left\{ \frac{\partial g}{\partial t}(\tilde{v}, t) + \alpha \frac{\partial}{\partial \tilde{v}}(\tilde{v} g(\tilde{v}, t)) \right\}, \end{aligned} \tag{34}$$

while

$$\begin{aligned} Q_p(f, f)(v) &= \int_{\mathbb{R}} dw \int_0^{2\pi} \frac{d\theta}{2\pi} [\chi^{-1} f(v^{**})\, f(w^{**}) - f(v)\, f(w)] \\ &= \int_{\mathbb{R}} d\tilde{w} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\alpha t} [\chi^{-1} g(\tilde{v}^{**})\, g(\tilde{w}^{**}) - g(\tilde{v})\, g(\tilde{w})] \\ &= e^{\alpha t} Q_p(g, g)(\tilde{v}). \end{aligned} \tag{35}$$

Thus the kinetic equation for g reads

$$\frac{\partial g(\tilde{v}, t)}{\partial t} + \alpha \frac{\partial(\tilde{v} g)}{\partial \tilde{v}} = Q_p(g, g)(\tilde{v}). \tag{36}$$

Note that, if g_∞ is a stationary solution of (36), from (32) it follows that

$$f_\infty(v) = e^{\alpha t} g_\infty(e^{\alpha t} v, t) \tag{37}$$

is a similarity solution of (1). Let us look for stationary solutions of (36), which solve

$$\alpha \frac{\partial}{\partial \tilde{v}}(\tilde{v} g(\tilde{v}, t)) = Q_p(g, g)(\tilde{v}, t). \tag{38}$$

As remarked in Section 3, the analysis of (38) is simpler by using Fourier transform, which reads

$$-\alpha\xi \frac{\partial \hat{g}}{\partial \xi} = \hat{Q}_p(\hat{g}, \hat{g})(\xi). \tag{39}$$

For any given $p > 0$, we look then for solutions to the equation

$$-\alpha\xi \frac{\partial \hat{g}}{\partial \xi} + \hat{g}(\xi) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, \hat{g}(\xi_p^+)\, \hat{g}(\xi_p^-). \tag{40}$$

By equality (12), it follows that, for all $\sigma > 0$

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\theta}{2\pi} (1 + \sigma\, |\xi_p^+|^{2/(1+p)})(1 + \sigma\, |\xi_p^-|^{2/(1+p)}) \\
&\quad = 1 + \sigma\, |\xi|^{2/(1+p)} + \sigma^2\, |\xi|^{4/(1+p)} \int_0^{2\pi} \frac{d\theta}{2\pi} \sin^2\theta \cos^2\theta \\
&\quad = 1 + \sigma\, |\xi|^{2/(1+p)} + \frac{\sigma^2}{8}\, |\xi|^{4/(1+p)}.
\end{aligned}$$

Hence, if

$$\hat{g}_\infty(\xi) = (1 + \sigma\, |\xi|^{2/(1+p)}) \exp\left\{ -\sigma\, |\xi|^{\frac{2}{1+p}} \right\}, \tag{41}$$

it follows that

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\, \hat{g}_\infty(\xi_p^+)\, \hat{g}_\infty(\xi_p^-) = \left(1 + \sigma\, |\xi|^{2/(1+p)} + \frac{\sigma^2}{8}\, |\xi|^{4/(1+p)} \right) \exp\left\{ -\sigma\, |\xi|^{\frac{2}{1+p}} \right\}.$$

Since

$$-\alpha\xi \frac{\partial \hat{g}_\infty}{\partial \xi} + \hat{g}_\infty(\xi) = \left(1 + \sigma\, |\xi|^{2/(1+p)} + \frac{2\alpha\sigma^2}{1+p}\, |\xi|^{4/(1+p)} \right) \exp\left\{ -\sigma\, |\xi|^{\frac{2}{1+p}} \right\},$$

$\hat{g}_\infty$ solves (40) if $\alpha = (1+p)/16$.

Now, the question is to understand if $\hat{g}_{\infty,p}$ is the Fourier transform of a probability density. By expanding it in Taylor series, one immediately comes to the conclusion that, if $p < 1$, $\hat{g}_{\infty,p}$ can not be the Fourier transform of a nonnegative function. In fact,

$$\hat{g}_{\infty,p}(\xi) = 1 - \sigma^2\, |\xi|^{4/(1+p)} + o(|\xi|^{4/(1+p)}),$$

where $4/(1+p) > 2$. Thus, the second moment of $g_\infty(v)$ has to be equal to zero, which implies at once that $g_{\infty,p}(v)$ is degenerate. A completely different situation appears when $p \geqslant 3$. In this range of p, $\hat{g}_{\infty,p}(\xi)$ is convex and satisfies all the conditions of Polya theorem (ref. 22, p. 168). Then $\hat{g}_{\infty,p}(\xi)$ is the characteristic function of an absolutely continuous distribution function. We are not able to draw similar conclusions for $1 < p < 3$, even if we conjecture that in this range of the dissipation parameter $\hat{g}_{\infty,p}(\xi)$ is a characteristic function. We proved

Theorem 7.1. For all $p > 0$ the dissipative Kac equation (1) possesses a exact solution $f_{\infty,p}(v,t)$, defined by

$$f_{\infty,p}(v,t) = \exp\left\{\frac{1+p}{16}\,t\right\} g_{\infty,p}\left(\exp\left\{\frac{1+p}{16}\,t\right\} v, t\right), \tag{42}$$

where $\hat{g}_{\infty,p}(\xi)$ is given by (41). If $p \geqslant 3$ or $p = 1$, $f_{\infty,p}(v,t)$ is nonnegative.

The domain of attraction of these exact solutions (when $p \geqslant 3$) can be easily recovered owing to the same method we used in Section 6 to study the convergence towards the steady states M_s. We obtain

Theorem 7.2. Let $p \geqslant 3$, and let $f(t)$ be the unique solution of the Kac equation (1), corresponding to the initial density f_0 such that, for some $0 < \delta \leqslant \frac{p-1}{p+1}$

$$\int |v|^{2s+\delta}\, |f_0(v) - g_{\infty,p}(v)|\, dv < \infty,$$

where $s = 2/(1+p)$. Then, $f(t)$ converges exponentially fast in Fourier metric towards equilibrium, and the following bound holds

$$d_{2s+\delta}(f(t), f_{\infty,p}(t)) \leqslant d_{2s+\delta}(f_0, g_{\infty,p}) \exp\{-(1-B_{p,\delta})\, t\} \tag{43}$$

where

$$B_{p,\delta} = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(|\sin\theta|^{4+(1+p)\delta} + |\cos\theta|^{4+(1+p)\delta}\right) < 1.$$

The value $p = 1$ separates from the others, because of the fact that only in this case the temperature of the steady state is finite. We remark in fact that, while for $p > 1$ the temperature of the steady state is zero, when $p < 1$ its temperature is unbounded. Let us set $\sigma = 1$. When $p = 1$, we can easily recover the form of the steady state in the velocity space, since $\hat{g}_{\infty,1}(\xi)$ satisfies the equality

$$\hat{g}''_{\infty,1}(\xi) = (-1+|\xi|)\, e^{-|\xi|} = -2e^{-|\xi|} + \hat{g}_{\infty,1}(\xi). \tag{44}$$

Now, recalling that $e^{-|\xi|}$ is the Fourier transform of the Cauchy density $(\pi(1+v^2))^{-1}$, (44) implies

$$(1+v^2)\, g_{\infty,1}(v) = \frac{2}{\pi(1+v^2)},$$

that is

$$g_{\infty,1}(v) = \frac{2}{\pi(1+v^2)^2}. \tag{45}$$

The density $g_{\infty,1}(v)$ has been found before as similarity solution of the Ulam model (see ref. 1). We can improve in this case the result of Theorem 7.2, by showing for $p=1$ the Ernst–Brito conjecture for the inelastic Kac model. The stationary solution $g_{\infty,1}$ possesses moments of order $2+\lambda$, for $0<\lambda<1$. Let us consider two initial densities f_0 and h_0 such that, for some $\lambda>0$,

$$\int_{\mathcal{R}} |v|^{2+\lambda} f_0(v)\, dv < +\infty, \qquad \int_{\mathcal{R}} |v|^{2+\lambda} h_0(v)\, dv < +\infty.$$

Without loss of generality, suppose that $\lambda<1$. Consider now the metric (24) defined in Section 5, and let us proceed as in Section 6. For any ξ it results

$$\frac{|\hat{f}(\xi,t)-\hat{h}(\xi,t)|}{|\xi|^{2+\lambda}} \leqslant \exp\{-(1-A_{1,\lambda})\, t\} \tag{46}$$

where $A_{1,\lambda}$ is defined as

$$\begin{aligned} A_{1,\lambda} &= \int_0^{2\pi} \frac{d\theta}{2\pi}\, [|\xi_1^+|^{2+\lambda} + |\xi_1^-|^{2+\lambda}] \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi}\, [|\cos\theta|^{2(2+\lambda)} + |\sin\theta|^{2(2+\lambda)}]. \end{aligned} \tag{47}$$

Remind now that the rescaled distribution g has been defined trough the change of variables $\tilde{v} = ve^{-\alpha t}$. If $p=1$, $\alpha = 1/8$, which implies $\tilde{v} = \frac{v}{\sqrt{T(t)}}$.

For the Fourier variable ξ, turns into $\tilde{\xi} = \xi \sqrt{T}(t)$. Let us define $\tilde{f}(v, t) = \sqrt{T(t)}\, f(\sqrt{T(t)}\, v, t)$. Inserting the scaling into (46), we get

$$d_{2+\lambda}(\tilde{f}(v, t), g_{\infty, 1}(v)) \leqslant d_{2+\lambda}(f_0, g_{\infty, 1}) \exp\{-[(1-A_{1,\lambda})-(2+\lambda)/8]\, t\}.$$

It turns out that exponential convergence to g_∞ occurs as soon as

$$G(\lambda) = \frac{2+\lambda}{8} + A_{1,\lambda} - 1 < 0 \tag{48}$$

i.e., more explicitly, when

$$\frac{2+\lambda}{2}\left\{1-\int_0^{2\pi}\frac{d\theta}{2\pi}\,[|\cos\theta|^4+|\sin\theta|^4]\right\}+\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\sin\theta|^{2(2+\lambda)}+|\cos\theta|^{2(2+\lambda)}) < 1. \tag{49}$$

It is clear that $G(0) = 0$. Moreover, with a direct computation we obtain $G(1) = 0$. Owing to expression (49) we recover that $G(\lambda)$ is convex. Hence we can conclude that $G(\lambda) < 0$ for any $0 < \lambda < 1$. For such a choice of λ,

$$\left\| \hat{f}\left(\frac{\xi}{\sqrt{T}}, t\right) - \hat{g}_{\infty, 1}(\xi) \right\|_{2+\lambda} \to 0 \tag{50}$$

exponentially with respect to time.

Theorem 7.3. Let $p = 1$, and let $g_{\infty, 1}(v)$ be given by (45). Let $f(v, t)$ be the solution of the Kac equation (1), corresponding to the initial density f_0 such that, for some $0 < \lambda < 1$

$$\int |v|^{2+\lambda} f_0(v)\, dv < \infty.$$

Then, $\tilde{f}(v, t) = \sqrt{T(t)}\, f(\sqrt{T(t)}\, v, t)$ converges exponentially fast in Fourier metric towards equilibrium, and the following bound holds

$$d_{2+\lambda}(\tilde{f}(t), g_{\infty, 1}) \leqslant d_{2+\lambda}(f_0, g_{\infty, 1}) \exp\{-|G(\lambda)|\, t\} \tag{51}$$

where $G(\lambda)$ is given by (48).

Theorem 7.3 is nothing but the Ernst–Brito conjecture for the case $p = 1$. In fact, the steady state $g_{\infty, 1}(v)$ of unit temperature attracts all then (rescaled) solutions to the inelastic Kac equation which initially have some moment higher than two which is initially bounded. In the next section we will discuss the possibility to verify the same conjecture for $p \neq 1$.

8. SIMILARITY SOLUTIONS OF BOUNDED TEMPERATURE

In Section 7 we studied the existence of steady states of Eq. (36), where g was obtained from f via (32). We found explicit stationary solutions to (36) provided $\alpha = (1+p)/16$. Unlikely, in general these solutions do not have bounded temperature. For this reason, in what follows we look for the existence of stationary solutions to (36) under the assumption $\alpha = \gamma/2$, with

$$\gamma = \gamma_p = 1 - \int_0^{2\pi} \frac{d\theta}{2\pi} \left(|\sin\theta|^{2(1+p)} + |\cos\theta|^{2(1+p)}\right). \tag{52}$$

Within this choice of α, and assuming that stationary solutions exist, by (33) their temperature is finite. If $p = 1$, we found that a stationary solution with bounded temperature exists, and that this solution satisfies in addition

$$\int |v|^\delta \, g_{\infty,1}(v)\, dv < \infty, \qquad \delta < 3.$$

Let us suppose that a stationary solution to (36) with bounded temperature exists, and let us investigate wether or not this solution has some moment of order greater than two which is bounded. This can be done by studying the behavior of the stationary solution $\hat{g}$ in a neighbor of $\xi = 0$. Because of the requirement that the temperature has to be finite, we assume that for small ξ the following expansion holds

$$\hat{g}_{\infty,p}(|\xi|) = 1 - \xi^2 + A\,|\xi|^a + o(|\xi|^a). \tag{53}$$

We now insert this expression in (40) and check wether the resulting equation admits a solution for the exponent a. We obtain

$$\begin{aligned} 1 - \xi^2 + A\,|\xi|^a - \frac{\gamma}{2}\{-2\xi^2 + Aa\,|\xi|^a\} + o(|\xi^a|) \\ = 1 - \xi^2 \int_0^{2\pi} \frac{d\theta}{2\pi}\,[|\cos\theta|^{2+2p} + \sin\theta^{2+2p}] \\ + A\,|\xi|^a \int_0^{2\pi} [|\cos\theta|^{(1+p)\,a} + \sin\theta^{(1+p)\,a}] + o(|\xi^a|). \end{aligned} \tag{54}$$

Summing up with respect to powers of ξ, we have

$$-(1-\gamma)\,\xi^2+A\left(1-\frac{a\gamma}{2}\right)|\xi|^a+\cdots$$

$$=-\xi^2\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\cos\theta|^{2+2p}+|\sin\theta|^{2+2p})$$

$$+A\,|\xi|^a\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\cos\theta|^{(1+p)\,a}+|\sin\theta|^{(1+p)\,a})+\cdots. \tag{55}$$

Since the coefficients of $|\xi^2|$ are automatically equal by definition of γ, it turns out that equality occurs if

$$1-\frac{a\gamma}{2}=\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\cos\theta|^{(1+p)\,a}+|\sin\theta|^{(1+p)\,a}). \tag{56}$$

We now focus on Eq. (56) and prove the following

Lemma 8.1. Let $G_p(\lambda)$ be the function defined as

$$G_p(\lambda)=\frac{2+\lambda}{2}\,\gamma+\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\cos\theta|^{(1+p)(2+\lambda)}+|\sin\theta|^{(1+p)(2+\lambda)})-1. \tag{57}$$

Then, there exists $\bar{p}>1$ such that, if $p\geqslant\bar{p}$, $G_p(\lambda)=0$, only for $\lambda_1=0$, while if $p<\bar{p}$, $G_p(\lambda)=0$ for $\lambda_1=0$ and $\lambda_2>0$. In this case $G_p(\lambda)<0$ for any $\lambda\in(0,\lambda_2)$.

Proof. By definition (7) we get

$$G_p(\lambda)=\frac{2+\lambda}{2}\left\{1-\int_0^{2\pi}\frac{d\theta}{2\pi}\,[|\cos\theta|^{2+2p}+|\sin\theta|^{2+2p}]\right\}$$

$$+\int_0^{2\pi}\frac{d\theta}{2\pi}\,(|\sin\theta|^{(1+p)(2+\lambda)}+|\cos\theta|^{(1+p)(2+\lambda)})-1$$

$$=\frac{2+\lambda}{2}\left\{1-2\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{2+2p}\right\}+2\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)(2+\lambda)}-1.$$

Let us evaluate the first and the second derivative of $G_p(\lambda)$.

$$G'_p(\lambda)=\frac{1}{2}\left\{1-2\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{2+2p}\right\}$$

$$+2(1+p)\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)(2+\lambda)}\log|\sin\theta| \tag{58}$$

and

$$G_p''(\lambda) = 2(1+p)^2 \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{(1+p)(2+\lambda)} [\log|\sin\theta|]^2. \tag{59}$$

It follows that $G_p(\lambda)$ is a convex function on its domain since $G_p''(\lambda) > 0$ for every $\lambda \geqslant 0$. It is easy to see that $\lambda = 0$ is a solution of (56). On the other hand, due to the convexity and to the fact that $\lim_{\lambda\to+\infty} G_p(\lambda) = +\infty$, there exists $\lambda_2 > 0$ such that $G_p(\lambda_2) = 0$ any time $G_p'(\lambda = 0) < 0$. Now

$$\begin{aligned} G_p'(\lambda = 0) &= \frac{1}{2} - \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{2+2p} + 2(1+p)\int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{2+2p} \log|\sin\theta| \\ &= \frac{1}{2} - \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{2+2p} (1 - \log|\sin\theta|^{2+2p}). \end{aligned} \tag{60}$$

Note that the integral decreases with respect to p, while it is strictly bigger than $1/2$ if $p = 0$. This ensures that there exists exactly one value $p = \bar{p}$ such that

$$\frac{1}{2} - \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{2+2\bar{p}} (1 - \log|\sin\theta|^{2+2\bar{p}}) = 0. \tag{61}$$

Then, we can conclude that $\bar{p} > 1$, since we showed in the previous section that $\lambda_2 = 1$ if $p = 1$.

Gathering all these computations, we can conclude that, if $p < \bar{p}$, there exists $\lambda_2 > 0$ such that $G_p(\lambda_2) = 0$. ∎

The importance of Lemma 8.1 is related both to the existence of a stationary solution to equation

$$\frac{\partial g}{\partial t} - \frac{\gamma_p}{2} \frac{\partial}{\partial v}(v g(v)) = Q_p(g, g)(v), \tag{62}$$

to the domain of attraction of the solution itself, and to the evolution of moments for the solution to Eq. (62).

9. EVOLUTION OF MOMENTS AND CONVERGENCE

We will first discuss the evolution of moments for the solution to Eq. (62). By construction, the temperature of $g(v, t)$ is constant in time, and equal to 1 thanks to the normalization conditions (5). We can use the computations leading to (6), choosing $\varphi(v)=|v|^{2+\lambda}$, where for the moment the positive constant $\lambda\leqslant 1$. For the sake of simplicity, we set $\gamma=\gamma_p$. Suppose that the initial density $g_0(v)=f_0(v)$ is such that

$$\int_{\mathbb{R}} |v|^{2+\lambda}\, g_0(v)\, dv = m_\lambda < \infty. \tag{63}$$

Then, since the contribution due to the term $\frac{\gamma}{2}\frac{\partial}{\partial v}(vg(v))$ can be evaluated integrating by parts,

$$\int_{\mathbb{R}} |v|^{2+\lambda}\frac{\gamma}{2}\frac{\partial}{\partial v}(vg(v))\, dv = -\frac{\gamma}{2}(2+\lambda)\int_{\mathbb{R}} |v|^{2+\lambda}\, g(v,t)\, dv,$$

we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} |v|^{2+\lambda}\, g(v,t)\, dv = \left(-1+\frac{\gamma}{2}(2+\lambda)\right)\int_{\mathbb{R}} |v|^{2+\lambda}\, g(v,t)\, dv \tag{64}$$

$$+\int_{\mathbb{R}^2\times[0,2\pi]}\frac{d\theta}{2\pi}\, dv\, dw\, |v_p^*|^{2+\lambda}\, g(v)\, g(w). \tag{65}$$

Let us recover a suitable upper bound for the last integral in (65). Owing to definition (4)

$$|v_p^*| = |v\cos\theta\, |\cos\theta|^p - w\sin\theta\, |\sin\theta|^p| \leqslant |v|\, |\cos\theta|^{1+p}+|w|\, |\sin\theta|^{1+p}.$$

On the other hand, given any two constants a, b, and $0<\lambda\leqslant 1$ the following inequality holds

$$(|a|+|b|)^\lambda \leqslant |a|^\lambda+|b|^\lambda. \tag{66}$$

Hence, choosing $a=|v|\,|\cos\theta|^{1+p}$ and $b=|w|\,|\sin\theta|^{1+p}$,

$$|v_p^*|^{2+\lambda}\leqslant (|a|+|b|)^2\,(|a|^\lambda+|b|^\lambda).$$

Now, consider that

$$\int_{\mathbb{R}^2\times[0,2\pi]}\frac{d\theta}{2\pi}\, dv\, dw(|a|^{2+\lambda}+|b|^{2+\lambda})\, g(v)\, g(w)$$

$$=\left(\int_0^{2\pi}\frac{d\theta}{2\pi}(|\sin\theta|^{(1+p)(2+\lambda)}+|\cos\theta|^{(1+p)(2+\lambda)}\right)\int_{\mathbb{R}} |v|^{2+\lambda}\, g(v,t)\, dv, \tag{67}$$

while

$$\begin{aligned}
&\int_{\mathbb{R}^2\times[0,2\pi]} \frac{d\theta}{2\pi}\,dv\,dw\,|a|^{1+\lambda}\,|b|\,g(v)\,g(w)\\
&\quad=\int_{\mathbb{R}^2\times[0,2\pi]} \frac{d\theta}{2\pi}\,dv\,dw\,|a|\,|b|^{1+\lambda}\,g(v)\,g(w)\\
&\quad=\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)(1+\lambda)}\,|\cos\theta|^{(1+p)}\int_{\mathbb{R}}|v|^{1+\lambda}\,g(v)\,dv\int_{\mathbb{R}}|w|\,g(w)\,dw\\
&\quad\leqslant\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)(1+\lambda)}\,|\cos\theta|^{(1+p)}.
\end{aligned}\tag{68}$$

In fact, since $\lambda \leqslant 1$ Hölder inequality and conservation of the temperature (5) imply

$$\begin{aligned}
&\int_{\mathbb{R}}|v|^{1+\lambda}\,g(v)\,dv\int_{\mathbb{R}}|w|\,g(w)\,dw\\
&\qquad\leqslant\left(\int_{\mathbb{R}}|v|^2\,g(v)\,dv\right)^{(1+\lambda)/2}\left(\int_{\mathbb{R}}|v|^2\,g(v)\,dv\right)^{1/2}=1.
\end{aligned}$$

Using the same ideas, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2\times[0,2\pi]} \frac{d\theta}{2\pi}\,dv\,dw\,|a|^{\lambda}\,|b|^2\,g(v)\,g(w)\\
&\quad=\int_{\mathbb{R}^2\times[0,2\pi]} \frac{d\theta}{2\pi}\,dv\,dw\,|a|^2\,|b|^{\lambda}\,g(v)\,g(w)\\
&\quad=\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)\lambda}\,|\cos\theta|^{2}\int_{\mathbb{R}}|v|^{\lambda}\,g(v)\,dv\int_{\mathbb{R}}|w|^2\,g(w)\,dw\\
&\quad\leqslant\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\sin\theta|^{(1+p)\lambda}\,|\cos\theta|^{2}.
\end{aligned}\tag{69}$$

Grouping all these inequalities, and recalling the expression of $G_p(\lambda)$ given by (57) we obtain the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}}|v|^{2+\lambda}\,g(v,t)\,dv\leqslant G_p(\lambda)\int_{\mathbb{R}}|v|^{2+\lambda}\,g(v,t)\,dv+B_{p,\lambda},\tag{70}$$

where

$$B_{p,\lambda} = 4 \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{(1+p)(1+\lambda)} |\cos\theta|^{(1+p)} + 2 \int_0^{2\pi} \frac{d\theta}{2\pi} |\sin\theta|^{(1+p)\lambda} |\cos\theta|^2. \tag{71}$$

By Lemma 8.1, provided $p < \bar{p}$, for any $\lambda < \lambda_2$, $G_p(\lambda) < 0$. In this case, inequality (70) gives un upper bound for the moment, that reads

$$\int_{\mathbb{R}} |v|^{2+\lambda} g(v, t)\, dv \leqslant m_\lambda + \frac{B_{p,\lambda}}{|G_p(\lambda)|} < \infty. \tag{72}$$

In the case $\lambda_2 > 3$ we can easily iterate our procedure to obtain that any moment of order $2+\lambda$, with $\lambda < \lambda_2$ which is bounded initially, remains bounded at any subsequent time. The only difference now is that the explicit expression of the bound is more and more involved.

If $p < \bar{p}$, we can immediately draw conclusions on the large-time convergence of class of probability densities $\{g(v,t)\}_{t \geqslant 0}$, By virtue of Prokhorov theorem (cf. ref. 22) the existence of a uniform bound on moments implies that this class is tight, so that any sequence $\{g(v, t_n)\}_{n \geqslant 0}$ contains an infinite subsequence which converges weakly to some probability measure $g_{\infty,p}$. Thanks to the previous result, provided $p < \bar{p}$, $g_{\infty,p}$ possesses moments of order $2+\lambda$, for $0 < \lambda < \lambda_2$.

It is now immediate to show that this limit is unique. To this aim, let us consider two initial densities $f_0(v)$ and $h_0(v)$ such that, for some $0 < \lambda < \lambda_2$,

$$\int_{\mathcal{R}} |v|^{2+\lambda} f_0(v)\, dv < +\infty, \qquad \int_{\mathcal{R}} |v|^{2+\lambda} h_0(v)\, dv < +\infty.$$

Proceeding as in the proof of Theorem 7.3, for any ξ we obtain

$$\frac{|\hat{f}(\xi, t) - \hat{h}(\xi, t)|}{|\xi|^{2+\lambda}} \leqslant \exp\{-(1 - A_{p,\lambda})\, t\} \tag{73}$$

where $A_{p,\lambda}$ is defined as

$$A_{p,\lambda} = \int_0^{2\pi} \frac{d\theta}{2\pi} [|\xi_p^+|^{2+\lambda} + |\xi_p^-|^{2+\lambda}]$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} [|\cos\theta|^{(1+p)(2+\lambda)} + |\sin\theta|^{(1+p)(2+\lambda)}].$$

Remind now that, since $\tilde{v} = v/\sqrt{T(t)}$, g has temperature equal to 1. Therefore, if we define $\tilde{f}(v,t) = \sqrt{T(t)}\, f(\sqrt{T(t)}\, v, t)$. Inserting the scaling into (46), we get

$$d_{2+\lambda}(\tilde{f}(v,t), \tilde{h}(v,t)) \leqslant d_{2+\lambda}(f_0, h_0) \exp\left\{-\left[(1-A_{p,\lambda}) - \frac{2+\lambda}{2}\gamma\right] t\right\}$$
$$= d_{2+\lambda}(f_0, h_0) \exp\{G_p(\lambda)\, t\}. \tag{74}$$

It turns out that the $d_{2+\lambda}$-distance between subsequences converges to zero as soon as $G_p(\lambda) < 0$, which is true, after Lemma 8.1 for all $0 < \lambda < \lambda_2$. For such a choice of λ,

$$\left\| \hat{f}\left(\frac{\xi}{\sqrt{T}}, t\right) - \hat{g}_{\infty,p}(\xi) \right\|_{2+\lambda} \to 0 \tag{75}$$

exponentially with respect to time.

We can now show that the limit function $g_{\infty,p}(v)$ is a stationary solution to (62). Thanks to the previous computations, we know that if condition (63) holds, both the solution $g(v,t)$ to Eq. (62) and $g_{\infty,p}(v)$ have moments of order $2+\bar{\lambda}$, with $0 < \bar{\lambda} < \lambda$ uniformly bounded. Hence, for any $t \geqslant 0$, proceeding as in the proof of Theorem 6.2, we obtain

$$d_{2+\bar{\lambda}}(Q_p(g(t), g(t)), Q_p(g_{\infty,p}, g_{\infty,p})) \leqslant (1 + A_{p,\lambda})\, d_{2+\bar{\lambda}}(g(t), g_{\infty,p}). \tag{76}$$

This implies the weak* convergence of $Q_p(g(t), g(t))$ towards $Q_p(g_{\infty,p}, g_{\infty,p})$. In particular, due to the equivalence among different metrics which metricize the weak* convergence of measures,[17, 26] if $C_0^1(\mathbb{R})$ denotes the set of compactly supported continuously differentiable functions, endowed with its natural norm $\|\cdot\|_1$, for all $\phi \in C_0^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(v)\, Q_p(g(t), g(t))(v)\, dv \to \int_{\mathbb{R}} \phi(v)\, Q_p(g_{\infty,p}, g_{\infty,p})(v)\, dv. \tag{77}$$

On the other hand, for all $\phi \in C_0^1(\mathbb{R})$, integration by parts gives

$$\int_{\mathbb{R}} \phi(v) \frac{\partial}{\partial v}(vg(v,t))\, dv = -\int_{\mathbb{R}} v\phi'(v)\, g(v,t)\, dv. \tag{78}$$

Since $|v\phi'(v)| \leqslant |v|\, \|\phi'\|_1$, and the temperature of $g(v,t)$ is equal to unity, the convergence of $d_{2+\bar{\lambda}}(g(t), g_{\infty,p})$ to zero implies

$$\int_{\mathbb{R}} v\phi'(v)\, g(v,t)\, dv \to \int_{\mathbb{R}} v\phi'(v)\, g_{\infty,p}(v)\, dv. \tag{79}$$

Finally, for all $\phi \in C_0^1(\mathbb{R})$ it holds

$$\int_{\mathbb{R}} \phi(v) \left\{ \frac{\partial}{\partial v} (v g_{\infty, p}(v)) - Q_p(g_{\infty, p}, g_{\infty, p})(v) \right\} dv = 0. \tag{80}$$

This shows that $g_{\infty, p}$ is the unique stationary solution to (62). We have

Theorem 9.1. Let $p < \bar{p}$, where $\bar{p}$ solves (61) and let $g_{\infty, p}(v)$ be the unique stationary solution to Eq. (62). Let $f(v, t)$ be the solution of the Kac equation (1), corresponding to the initial density f_0 such that, for some $0 < \lambda < \lambda_2$

$$\int |v|^{2+\lambda} f_0(v)\, dv < \infty.$$

Then, $\tilde{f}(v, t) = \sqrt{T(t)}\, f(\sqrt{T(t)}\, v, t)$ satisfies

$$\int |v|^{2+\lambda} \tilde{f}(v, t)\, dv \leqslant c_{p, \lambda} < \infty.$$

If $0 < \lambda \leqslant 1$ the constant $c_{p, \lambda}$ is given by (72). Moreover, $\tilde{f}(v, t)$ converges exponentially fast in Fourier metric towards $g_{\infty, p}(v)$, and the following bound holds

$$d_{2+\lambda}(\tilde{f}(t), g_{\infty, p}) \leqslant d_{2+\lambda}(f_0, g_{\infty, p}) \exp\{-|G_p(\lambda)|\, t\} \tag{81}$$

where $G_p(\lambda)$ is given by (57).

Theorem 9.1 is nothing but the Ernst–Brito conjecture for the case $p < \bar{p}$.

10. CONCLUSIONS

In this paper we investigated the large-time behavior of a dissipative kinetic model obtained by generalizing the classical model known as Kac caricature of a Maxwell gas. We found connections between the cooling problem for this model and the classical central limit theorem for stable laws. These connections enabled us to make use of arguments typical of probability theory. The existence of stationary solutions with infinite energy is known also for the standard inelastic Maxwell model.[8] These solutions, however, do not look like stable laws. In analogy to the model discussed in this paper, we conjecture that these stationary solutions possess a domain of attraction, and it would be highly interesting to investigate on the existence of a central limit theorem.

Concerning the question posed by Ernst and Brito in ref. 16, we showed that this conjecture is true in the range of low dissipation given by $p < \bar{p}$, characterized by the fundamental fact that some moment of order greater than two for the rescaled Eq. (62) remains uniformly bounded in time. In this range, in fact, the limit distribution is shown to be unique, and its tails are completely determined in terms of the solution of a transcendental equation. We left open the problem of the convergence in the range of high dissipation $p \geqslant \bar{p}$. In this range of the parameter in fact it is not possible to conclude even with the existence of a limit distribution (without uniqueness) with unit temperature. To our knowledge, the existence of different regimes of dissipation for the same model which lead to completely different behaviors, has never been observed before, and certainly requires further investigations.

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REFERENCES

1. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Kinetic models of inelastic gases, *Math. Models Methods Appl. Sci.* **12**:965–983 (2002).
2. N. Ben-Naim and P. Krapivski, Multiscaling in inelastic collisions, *Phys. Rev. E* **61**: R5–R8 (2000).
3. A. V. Bobylev, The theory of the nonlinear spatially uniform Boltzmann equation for Maxwellian molecules, *Sov. Sci. Rev. C* **7**:111–233 (1988).
4. A. V. Bobylev, J. A. Carrillo, and I. Gamba, On some properties of kinetic and hydrodynamics equations for inelastic interactions, *J. Stat. Phys.* **98**:743–773 (2000).
5. A. V. Bobylev and C. Cercignani, Moment equation for a granular material in a thermal bath, *J. Stat. Phys.* **106**:547–567 (2002).
6. A. V. Bobylev and C. Cercignani, Exact eternal solutions of the Boltzmann equation, *J. Stat. Phys.* **106**:1019–1038 (2002).
7. A. V. Bobylev and C. Cercignani, Self-similar solutions of the Boltzmann equation and their applications, *J. Stat. Phys.* **106**:1039–1071 (2002).
8. A. V. Bobylev and C. Cercignani, Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions, *J. Stat. Phys.* **110**:333–375 (2003).
9. A. V. Bobylev, C. Cercignani, and G. Toscani, Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials, *J. Stat. Phys.* **111**: 403–417 (2003).

10. E. A. Carlen, E. Gabetta, and G. Toscani, Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas, *Comm. Math. Phys.* **305**:521–546 (1999).
11. E. A. Carlen, M. C. Carvalho, and E. Gabetta, Central limit theorem for Maxwellian molecules and truncation of the Wild expansion, *Comm. Pure Appl. Math.* **53**:370–397 (2000).
12. J. A. Carrillo, C. Cercignani, and I. M. Gamba, Steady states of a Boltzmann equation for driven granular media, *Phys. Rev. E* **62**:7700–7707 (2000).
13. C. Cercignani, Shear flow of a granular material, *J. Stat. Phys.* **102**:1407–1415 (2001).
14. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer Series in Applied Mathematical Sciences, Vol. 106 (Springer-Verlag, New York, 1994).
15. M. H. Ernst and R. Brito, High energy tails for inelastic Maxwell models, *Europhys. Lett.* **43**:497–502 (2002).
16. M. H. Ernst and R. Brito, Scaling solutions of inelastic Boltzmann equation with over-populated high energy tails, *J. Stat. Phys.* **109**:407–432 (2002).
17. E. Gabetta, G. Toscani, and B. Wennberg, Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation, *J. Stat. Phys.* **81**:901–934 (1995).
18. T. Goudon, S. Junca, and G. Toscani, Fourier-based distances and Berry–Esseen like inequalities for smooth densities, *Monatsh. Math.* **135**:115–136 (2002).
19. P. K. Haff, Grain flow as a fluid-mechanical phenomenon, *J. Fluid Mech.* **134**:401–430 (1983).
20. Hailiang Li and G. Toscani, Long-time asymptotics of kinetic models of granular flows, preprint (2002).
21. J. Hoffmann-Jørgensen, *Probability with a View Towards Statistics*, Vol. I (Chapman & Hall, New York, 1994).
22. R. G. Laha and V. K. Rohatgi, *Probability Theory* (Wiley, New York, 1979).
23. M. Kac, *Probability and Related Topics in the Physical Sciences* (Interscience, London/New York, 1959).
24. H. P. McKean, Jr., Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas, *Arch. Rational Mech. Anal.* **21**:343–367 (1966).
25. G. Toscani, One-dimensional kinetic models of granular flows, *RAIRO Modél Math. Anal. Numér.* **34**:1277–1292 (2000).
26. G. Toscani and C. Villani, Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas, *J. Stat. Phys.* **94**:619–637 (1999).
27. G. Toscani and C. Villani, Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation, *Comm. Math. Phys.* **203**:667–706 (1999).